# Multi-bubble blow-up solutions to stochastic nonlinear Schrödinger equations

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# Contents

# I. Motivations

We are concerned with the blow-up of stochastic nonlinear Schrödinger equation:

$$dX = i\Delta X dt + i|X|^{\frac{4}{d}} X dt - \mu X dt + X dW(t), \qquad \text{(SNLS)}$$
$$X(0) = X_0 \in H^1(\mathbb{R}^d).$$

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- $W$ is the colored Wiener process

$$W(t,x) = \sum_{k=1}^{N} i\phi_k(x) B_k(t), \quad x \in \mathbb{R}^d, \ t \geq 0.$$

  $\phi_k \in C_b^\infty(\mathbb{R}^d)$, $\mathrm{Re} W = 0$ (*conservative case*),
  $B_k$ are independent real-valued Brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

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- $XdW$ is taken in the sense of controlled rough path.
- $\mu(x) = \frac{1}{2} \sum_{j=1}^{N} \phi_k^2(x)$, $x \in \mathbb{R}^d$. (Physic motivation: conservation of mass.)

# I. Motivations: SNLS

**Physical motivations**

1. Propagation of nonlinear dispersive waves in random media (e.g. thermal vibration of molecules): $|X(t)|_2^2 = |X_0|_2^2$.
   [Bang et al.: Phys. Rev. 1994; Appl. Anal. 1995]

   Noise : random potential, approximated by Gaussian noise

2. Theory of measurements continuous in time in quantum mechanics.
   [Barchielli, Gregoratti: Lecture Notes Physics, 2009]

   Noise : stochastic continuous measurement in equation of state $X$

   $\widehat{P}_{X_0}^T(F) := \int_F |X(T)|_2^2 dP$ : physical probability of events occurring in $[0, T]$

   "Physical Probability Law", martingale property ($\mathbb{E}|X(t)|_2^2 = |X_0|_2^2$)

# I. Motivations: NLS

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(NLS) admits rich invariance and conservation laws:

- **Invariance**:
  Invariant under the translation, scaling, phase rotation, Galilean transform.
  In particular, $L^2$-norm is invariant under $u(t,x) \to \lambda^{-\frac{d}{2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ $\Rightarrow$ mass-critical.

  **Pseudo-conformal invariance**, particularly important in the mass-critical case, under

$$u(t,x) \to (-t)^{-\frac{d}{2}} u(\frac{1}{-t}, \frac{x}{-t}) e^{-i\frac{|x|^2}{4t}}, \qquad t \neq 0.$$

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- **Conservation law**:

$$\text{Mass}: \quad M(u)(t) := \int_{\mathbb{R}^d} |u(t)|^2 dx = M(u_0),$$

$$\text{Energy}: \quad E(u)(t) := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t)|^2 dx - \frac{d}{2d+4}\int_{\mathbb{R}^d} |u(t)|^{2+\frac{4}{d}} dx = E(u_0),$$

$$\text{Momentum}: \quad Mom(u) := \text{Im} \int_{\mathbb{R}^d} \nabla u \bar{u} dx = Mom(u_0).$$

# I. Motivations: NLS

An important role is played by the **ground state** $Q$:

which is the unique positive radial solution to the nonlinear elliptic equation

$$\Delta Q - Q + Q^{1+\frac{4}{d}} = 0.$$

**Threshold** ([Weinstein: CMP 1982]):

The mass of ground state is exactly the threshold for GWP and blow-up:

- Solutions exist globally in the subcritical mass case $\|u_0\|_{L^2} < \|Q\|_{L^2}$;
- Singularity can be formed in the (super)critical mass case $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$.

# I. Motivations: NLS

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1. Solitary wave $u(t,x) = Q(x)e^{it}$, exists globally, but does not scatter.
2. Pseudo-conformal blow-up solution:

$$S_T(t,x) := \frac{1}{(T-t)^{\frac{d}{2}}} Q\Big(\frac{x}{T-t}\Big) e^{\frac{i}{T-t} - i\frac{|x|^2}{4(T-t)}}, \quad x \in \mathbb{R}^d, \ t < T.$$

**Note:**

- $S_T$ is obtained from the solitary wave solution and pseudo-conformal symmetry.
- $S_T$ blows up at time $T$ with rate $(T-t)^{-1}$.
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- $\|S_T\|_{L^2} = \|Q\|_{L^2} \Rightarrow$ minimal mass blow-up solutions
- Instability.

# I. Motivations: NLS

**Rigidity result** ([Merle: Duke Math. J. 1993]):

The pseudo-conformal blow-up solution is exactly the unique minimal mass blow-up solution to (NLS), up to the invariance of (NLS).

**Construction results** for other dispersive equations:

- Mass-critical inhomogeneous Hartree equation [Cao, Su: JMP, 2013],
- Mass-citical gKdV equation [Martel, Merle, Raphaël: JEMS, 2017],
- Modified Benjamin-Ono equation [Martel, Pilod: Math. Ann., 2017].

# I. Motivations: NLS

**Super-critical mass case with mass slightly above** $\|Q\|_{L^2}$: $\|u_0\|_{L^2} = \|Q\|_{L^2} + \varepsilon$

Two different asymptotic scenarios have been observed:

1. the pseudo-conformal blow-up rate $\|\nabla u\|_{L^2} \sim \frac{1}{T-t}$;
2. the log-log blow-up rate $\|\nabla u\|_{L^2} \sim \sqrt{\frac{\log|\log(T-t)|}{T-t}}$.

See the series of works by Merle and Raphaël: [Merle, Raphaël: AOM, 2005; GFA, 2003; Invet. Math. 2004, JAMS, 2006].

See also the loglog blow-up solutions with $L^2$ initial data (low regularity): [Fan, Mendelson: arXiv: 2010.07821].

# I. Motivations: NLS

**Super-critical mass case with large mass**:

The complete characterization of singularity formation is open.

Mass quantization conjecture [Merle, Raphaël: CMP, 2005]:

General blow-up solutions shall concentrate the mass $m_k(\geq \|Q\|_{L^2}^2)$ at the singularities and converge strongly in $L^2$ to a residue $u^*$ away from the singularities.

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One important step is to construct **multi-bubble blow-up solutions**:

- Multi-bubble blow-up solutions with pseudo-conformal rate [Merle: CMP, 1990];
- Multi-bubble blow-up solutions with loglog rate [Fan: AIHP, 2017].

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**Question:** What about stochastic nonlinear Schrödinger equation ?

# I. Motivations: SNLS

**Stochastic case:**

1. Physical references: mass-critical case ($d = 1, 2$)
[Bang, Christiansen, If, Rasmussen: Phys. Rev. E, 1994; Appl. Anal. 1995],
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Two dimensional collapse is important in nonlinear optics.

2. Numerical study:
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3. Mass-supercritical case:
   [de Bouard, Debussche: PTRF, 2002; AOP, 2005].
   The *conservative* noise has the effect to accelerate blow-up with positive probability. The proof is based on a stochastic version of virial evolution initiated in [Glassey: JMP, 1977] and the support theorem.

4. Mass-(super)critical case:
   [Barbu, Röckner, Z.: JDE, 2016].
   The *non-conservative* noise has the damped effect to prevent blow-up with high probability.

# I. Motivations: SNLS

**Stochastic case:** Quantitative descriptions of blow-up dynamics

1. Critical mass case:
   [Su, Z.: arXiv: 2002.09659]
   Construction of minimal mass blow-up solutions

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**Goal**: Study multi-bubble blow-up solutions

1. Construction ?
2. Uniqueness ?

# II. Main results

**Assumptions**

- Asymptotical flatness (A0): For $1 \leq k \leq N$ and for any multi-index $\nu \neq 0$,

$$\lim_{|x| \to \infty} \langle x \rangle^2 |\partial_x^\nu \phi_k(x)| = 0.$$

- Flatness at the origin (A1): $\exists\ \nu_* \in \mathbb{N}$ such that for $1 \leq k \leq N$ and $1 \leq j \leq K$,

$$\partial_x^\nu \phi_k(x_j) = 0, \quad \forall\ 0 \leq |\nu| \leq \nu_*.$$

**Two cases**

- **Case (I):** $\{x_j\}_{j=1}^K$ are arbitrary distinct points in $\mathbb{R}^d$, and $\{\omega_j\}_{j=1}^K (\subseteq \mathbb{R}^+)$ satisfy that for some $\omega > 0$, $|\omega_j - \omega| \leq \varepsilon$ for any $1 \leq j \leq K$, where $\varepsilon > 0$;

- **Case (II):** $\{\omega_j\}_{j=1}^K$ are arbitrary points in $\mathbb{R}^+$, and $\{x_j\}_{j=1}^K (\subseteq \mathbb{R}^d)$ satisfy that $|x_j - x_l| \geq \varepsilon^{-1}$ for any $1 \leq j \neq l \leq K$, where $\varepsilon > 0$.

**Example:** The singularities $\{x_k\}$ can be arbitrary if $\{\omega_k\}$ are the same.

# II. Main results

**Theorem (Existence)**

Consider $d = 1, 2$. Assume $(A0)$ and $(A1)$ with $\nu_* \geq 5$. Let $K \in \mathbb{N}/\{0\}$, $\{\vartheta_j\} \subseteq \mathbb{R}$, $\{x_j\}_{j=1}^K \subseteq \mathbb{R}^d$ and $\{\omega_j\}_{j=1}^K \subseteq \mathbb{R}^+$ satisfy either $\mathrm{Case}$ (I) or $\mathrm{Case}$ (II).

Then, for $\mathbb{P} - a.e.$ $\omega$ there exists $\varepsilon^*(\omega) > 0$ small enough such that the following holds. For any $\varepsilon \in (0, \varepsilon^*]$, there exists $\tau^* > 0$ small enough such that for any $T \in (0, \tau^*(\omega)]$, there exist $X_0(\omega) \in \Sigma$ and a corresponding blow-up solution $X(\omega)$, satisfying

$$\|e^{-W(t,\omega)}X(t,\omega) - \sum_{j=1}^{K} S_j(t)\|_\Sigma \leq C(T-t)^{\frac{1}{2}(\nu_*-5)+\zeta}, \quad t \in [0,T),$$

where $\Sigma := \{v \in H^1 : |x|v \in L^2\}$, $C > 0$, $\zeta \in (0,1)$, and $S_j$, $1 \leq j \leq K$, are the pseudo-conformal blow-up solutions

$$S_j(t,x) = (\omega_j(T-t))^{-\frac{d}{2}} Q(\frac{x-x_j}{\omega_j(T-t)}) e^{-\frac{i}{4}\frac{|x-x_j|^2}{T-t} + \frac{i}{\omega_j^2(T-t)} + i\vartheta_j}, \quad t \in (0,T).$$

# II. Main results

**Remark**

- The constructed blow-up solution concentrates at the given $K$ points:

$$|X(t,\omega)|^2 \rightharpoonup \sum_{j=1}^{K} \|Q\|_{L^2}^2 \delta_{x=x_j}, \quad as \ t \to T.$$

- The asymptotic order can be improved if the noise is more flat at the singularities. If $\nu_* \geq 6$, the asymptotic can be even taken in the more regular space $H^{\frac{3}{2}}$.

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- Multi-bubble blow-up solutions were first constructed by Merle for NLS when $d \geq 1$, the main blow-up profile is built on any function that decay exponentially, while the frequencies $\{\omega_j\}_{j=1}^K$ are required to have a uniform positive lower bound and the asymptotic is taken in the space $L^{2+\frac{4}{d}}$.

  The above multi-bubble solutions are constructed when $d = 1, 2$, the blow-up profile is built on the ground state. The gain here is that, in $\mathrm{Case}\ (\mathrm{I})$ the frequencies are allowed to be arbitrarily small, and the asymptotic is taken in the energy space $H^1$, which is important to study the uniqueness problem.

# II. Main results

**Theorem (Conditional uniqueness)**

Consider $d = 1, 2$. Assume $(A0)$ and $(A1)$ with $\nu_* \geq 5$. Let $K \in \mathbb{N}$, $\{\vartheta_j\} \subseteq \mathbb{R}$, $\{x_j\}_{j=1}^K \subseteq \mathbb{R}^d$ and $\{\omega_j\}_{j=1}^K \subseteq \mathbb{R}^+$, satisfy $\mathrm{Case\ (I)}$ or $\mathrm{Case\ (II)}$.

Then, for $\mathbb{P} - a.e.\ \omega$ there exists $\varepsilon^*(\omega) > 0$ very small such that the following holds. For any $\varepsilon \in (0, \varepsilon^*]$, there exists $\tau^* > 0$ small enough such that for any $T \in (0, \tau^*(\omega)]$, there exists a unique blow-up solution $X(\omega)$ satisfying

$$\|e^{-W(t,\omega)} X(t,\omega) - \sum_{j=1}^{K} S_j(t)\|_{H^1} \leq C(T-t)^{3+\zeta}, \quad t \in [0, T),$$

where $C > 0$, $\zeta \in (0, 1)$, and $\{S_j\}$ are the pseudo-conformal blow-up solutions.

# II. Main results

**Application 1: minimal mass blow-up solutions**

**Theorem**

Consider $d = 1, 2$. Assume $(A0)$ and $(A1)$, $\nu_* \geq 5$. Let $x_*, \omega_*, \vartheta_*$ be any given points.

Then, for $\mathbb{P}$-a.e. $\omega$ there exists $\tau^*(\omega) > 0$ sufficiently small, such that for any $T \in (0, \tau^*(\omega)]$ there exists a minimal mass blow-up solution $X(\omega)$ satisfying

$$\|e^{-W(t,\omega)}X(t,\omega) - S_*(t)\|_\Sigma \leq C(T-t)^{\frac{1}{2}(\nu_*-5)+\zeta}, \quad t \in [0, T),$$

where $C > 0$, $\zeta \in (0, 1)$, $S_*$ is as above with $x_*, \omega_*, \vartheta_*$ replacing $\alpha_j, \omega_j, \vartheta_j$, respectively.

Moreover, in the case where $\nu_* \geq 11$, there exists a unique minimal mass blow-up solution $X(\omega)$ satisfying

$$\|e^{-W(t,\omega)}X(t,\omega) - S_*(t)\|_{H^1} \leq C(T-t)^{3+\zeta}, \quad t \in [0, T).$$

# II. Main results

**Application 2: NLS with lower perturbations**

$$i\partial_t v + \Delta v + |v|^{\frac{4}{d}} v + a_1 \cdot \nabla u + a_2 u = 0,$$

where

$$a_1(t,x) = 2i \sum_{k=1}^{N} \nabla \phi_k(x) h_k(t),$$

$$a_2(t,x) = -\sum_{j=1}^{d} (\sum_{k=1}^{N} \partial_j \phi_k(x) h_k(t))^2 + i \sum_{k=1}^{N} \Delta \phi_k(x) h_k(t),$$

$\phi_k$ satisfies Assumptions $(A0)$ and $(A1)$ and $h_k \in C(\mathbb{R}^+; \mathbb{R})$, $1 \leq k \leq K$.

In particular, if $h_k = B_k$, then the above equation is called RNLS.

These results are also applicable to NLS, in which $a_1, a_2 \equiv 0$.

**Note**: Absence of the pseudo-conformal invariance and the conservation law of energy.

# III. Further work

Further work: [Cao, Su, Zhang: arXiv: 2105.14554].

Consider the canonical focusing $L^2$-critical nonlinear Schrödinger equation (NLS):

$$i\partial_t u + \Delta u + |u|^{\frac{4}{d}} u = 0.$$

**Uniqueness Problem:** does there exist a unique multi-bubble blow-up solution $v$ to (NLS) such that

$$\|v(t) - \sum_{k=1}^{K} S_k(t)\|_{H^1} = o(1), \quad as\ t \to T. \qquad \big(o(1)\ Condition\big)$$

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**Remark:**

- Related to the open problem of uniqueness of multi-solitons [Martel: ICM 2018].
- Related to the solitary wave conjecture.
- Uniqueness is known in the single bubble case by the seminal paper [Merle: Duke Math. J., 1993]. The strong rigidity result was also proved by [Raphaël, Szeftel: JAMS, 2011] for inhomogeneous NLS.
- Uniqueness is open in the multi-bubble case. It provides an illustration of the rigidity of equation (NLS) around the pseudo-conformal blow-up solutions.

# III. Further work

**Remark:**

- Related to the *mass quantization conjecture* in [Merle, Raphaël: CMP, 2005]: it states that blow-up solutions to (NLS) shall concentrate the mass $m_k (\geq \|Q\|_{L^2}^2)$ at the singularities and converge strongly in $L^2$ to a residue $u^*$ away from the singularities. The above uniqueness problem provides one uniqueness class for such solutions.

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- Similarity with local uniqueness problem of bubbling solutions to nonlinear elliptic equations. [Cao, Peng, Yan: Cambridge University Press, 2021]

  For instance, the local uniqueness of multi-peak solutions $u_\varepsilon$ to Brezis-Nirenberg problem concentrating at different points $\{x_1, \dots, x_K\}$ satisfying

$$\|u_\varepsilon - \sum_{k=1}^{K} PU_{x_{k,\varepsilon},\lambda_{k,\varepsilon}}\|_{H^1} = o(1), \quad x_{k,\varepsilon} \to x_k, \ \textit{and} \ \lambda_{k,\varepsilon} \to +\infty, \quad \textit{as} \ \ \varepsilon \to 0,$$

  where $P$ is the projection from $H^1(\Omega)$ onto $H_0^1(\Omega)$, $\Omega$ is a smooth and bounded domain in $\mathbb{R}^d$, and $U_{x,\lambda}$ solves the elliptic equation $\Delta u + u^{\frac{d+2}{d-2}} = 0$ in $\mathbb{R}^d$, $d \geq 5$.

# III. Further work

**Theorem** (Multi-bubble blow-up case)

Consider (NLS) in dimensions $d = 1, 2$. Let $T \in \mathbb{R}$, $K \in \mathbb{N} \setminus \{0\}$. Let $\{\vartheta_k\} \subseteq \mathbb{R}$, $\{\omega_k\}$ and $\{x_k\}$ satisfy either $\mathrm{Case\ (I)}$ or $\mathrm{Case\ (II)}$.

Then, for any $\zeta \in (0, 1)$, there exists $\varepsilon^* > 0$ such that the following holds. For any $0 < \varepsilon < \varepsilon^*$, there exists a unique multi-bubble blow-up solution $v$ such that

$$\|v(t) - \sum_{k=1}^{K} S_k(t)\|_{L^2} + (T - t)\|\nabla v(t) - \sum_{k=1}^{K} \nabla S_k(t)\|_{L^2} = o(1), \quad as\ t\ close\ to\ T,$$

and additionally double average condition

$$\frac{1}{T-t}\int_t^T \frac{1}{T-s}\int_s^T \|v(r) - \sum_{k=1}^{K} S_k(r)\|_{H^1}^2 drds = \mathcal{O}((T-t)^{\zeta}), \quad (0+) - Condition$$

where $S_k$, $1 \le k \le K$, are the pseudo-conformal blow-up solutions.

# III. Further work

**Theorem (continued)**

Moreover, the unique solution $v$ converges exponentially fast to $\sum_{k=1}^{K} S_k$ in the pseudo-conformal space, i.e., there exists $\delta > 0$ such that

$$\|v(t) - \sum_{k=1}^{K} S_k(t)\|_{\Sigma} = \mathcal{O}(e^{-\frac{\delta}{T-t}}), \quad \textit{for } t \textit{ close to } T.$$

In particular, the above results hold for the multi-bubble blow-up solutions $v$ such that

$$\|v(t) - \sum_{k=1}^{K} S_k(t)\|_{H^1} = \mathcal{O}((T-t)^{\zeta}), \quad \textit{for } t \textit{ close to } T. \qquad (0+ \ \textit{Condition})$$

# Thank you